CURRENT VORTICES AND CRITICAL SURFACES IN MAGNETOHYDRODYNAMIC FLOWS

(TOKOVYE VIKHRI I KRITICHESKIE POVERKHNOSTI V MAGNITOGIDRODINAMICHESKOM POTOKE)

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This paper considers the steady axisymmetric flow of an ideally conducting plasma across an azimuthal magnetic field. The possible appearance of singular points (of elliptic and hyperbolic types) in the family of current lines $rH_\varphi = \mathrm{const}$ is investigated. The shapes of the critical surfaces are considered, on which the flow velocity v attains the signal velocity

$$c_s = \sqrt{c_T^2 + H_\varphi^2/4\pi\rho}$$

The stability of such flows is discussed.

1. Electric current lines. If the fluid velocity has two components v_r and v_z , and the magnetic field has only one component $H = H_\varphi$, then under the conditions of axial symmetry, the equations of magnetohydrodynamics for isentropic flow [$S(\xi)$ = const] reduce to the system [1]

$$\frac{1}{\rho r}\frac{\partial}{\partial r}\frac{1}{\rho r}\frac{\partial \xi}{\partial r} + \frac{1}{\rho r}\frac{\partial}{\partial z}\frac{1}{\rho r}\frac{\partial \xi}{\partial z} + \frac{\rho r^2}{2}\frac{dB^2}{d\xi} = \frac{dU}{d\xi} \qquad (1.1)$$

$$W(\rho) = {}^1/_2\, v^2 + \rho r^2 B^2 = U(\xi) \qquad (1.2)$$

Here ρ is the density, W the enthalpy, $\xi = \xi(r,z)$ the stream function which defines the velocity components

$$v_r = -\frac{1}{\rho r}\frac{\partial \xi}{\partial z}, \qquad v_z = \frac{1}{\rho r}\frac{\partial \xi}{\partial r} \qquad (1.3)$$

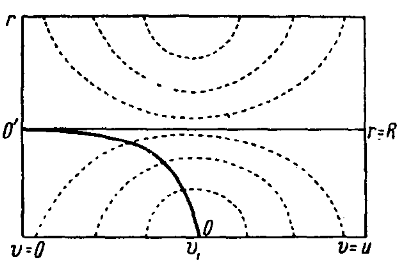

Fig. 1

The arbitrary functions $B(\xi)$ and $U(\xi)$ depend only on ξ . In addition to the Bernoulli integral (1.2), we have the "frozen" integral of force lines of the magnetic field.

$$I / \rho r^2 = B(\xi) \qquad (I \equiv rH_\varphi / \sqrt{4\pi}) \qquad (1.4)$$

The streamlines of the fluid are

defined by Equation $\xi(r,z)$ = const , while the electric current lines are given by Equation $I(r,z)$ = const

Let us consider the flow in the space between two electrodes (Fig.1). Plasma acceleration in this system occurs both due to thermal energy and action of electrodynamic forces. In the case when the magnetic pressure is small in comparison with the gas pressure, i.e.

$$\beta \equiv 8\pi p H^{-2} \gg 1 \tag{1.5}$$

the acceleration is mainly due to the thermal energy. In this case, the electric current aligns with the flow, which to first approximation is governed by the laws of ordinary gas dynamics; thus, it is capable of changing direction, being in one part of the flow accelerating and in another part decelerating. Consequently, when (1.5) is satisfied, one may expect the appearance of current eddies. K.V. Brushlinskii, N.I. Gerlakh and A.I. Morozov, by means of numerical computations with an electronic computer, discovered the phenomenon of the current eddies at electrodes: the current leaving one electrode returns to the same electrode before reaching the other electrode. Below, this phenomenon will be studied analytically.

We note that the current density component perpendicular to the streamline can vanish only at the point where

$$r\frac{dv^2}{ds} - 4\rho W'(\rho)\frac{dr}{ds} = 0 \tag{1.6}$$

This equation is obtained from the condition $j_\perp = r^{-1}\,\partial I / \partial s = 0$, if (1.2) and (1.4) are differentiated along the streamline $r = r(s)$ According to (1.6), when the flow is accelerating $dv^2/ds > 0$, the current may form loops only when $dr/ds > 0$, i.e. on the expanding part of the electrode.

To calculate the flow, we limit ourselves to the case $U(\xi)$ = const , and we assume that the magnetic pressure is small in comparison with the gas pressure ($\beta \gg 1$). Moreover, we shall consider the flow to be slowly varying along the z-axis. Neglecting in (1.1) and (1.2) quantities of order B^2, $(\partial\xi/\partial z)^2$ and $\partial^2\xi/\partial z^2$, we obtain

$$\frac{\partial}{\partial r}\frac{1}{\rho r}\frac{\partial \xi}{\partial r} = 0, \qquad W(\rho) + \frac{1}{2}\left(\frac{1}{\rho r}\frac{\partial\xi}{\partial r}\right)^2 = U \tag{1.7}$$

Integration of these equations yields

$$\xi = \frac{\rho V}{2}(r^2 - R^2), \qquad W(\rho) + \frac{1}{2}V^2 = U \tag{1.8}$$

Here $V(z)$ and $R(z)$ are arbitrary slowly varying functions of the variable z , the knowledge of which permits the determination of the corresponding streamlines $\xi(r,z)$ = const . From the second equation of (1.8), it follows that ρ is also dependent on z only. We set R = const , i.e we require that one of the streamlines be a straight line $r = R$.
The pattern of the streamlines is given in Fig.1, where the velocity $V(z)$ lies along the z-axis. The streamlines are equipotential lines and any pair of them may represent the electrodes. In the neighborhood of the planes

$z = z_0$ and $z = z_M$, where the velocity V equals zero and the maximum value $\sqrt{2U} \equiv u$, respectively, the flow cannot be slowly varying as required, and consequently, the results thus obtained are applicable only in the middle part of the nozzle, where the streamlines are still sufficiently smooth.

To determine the singular point in the family of current lines $I(r,z) =$ $=$ const, we set to zero the derivatives $\partial I/\partial r$ and $\partial I/\partial z$ of $I = \rho r^2 B(\xi)$; we get

$$\frac{\partial I}{\partial r} = \rho r\,[2B + \rho r^2 V B'] = 0, \quad \frac{\partial I}{\partial z} = r^2\left[\frac{d\rho}{dz}B + \rho\frac{r^2 - R^2}{2}\frac{d(\rho V)}{dz}B'\right] = 0 \qquad (1.9)$$

From this, it is clear that for singular point to exist, it is necessary that $B'/B < 0$, and in the isomagnetic case (B = const), no singular points exist. Eliminating from (1.9) the quantity $B'(\xi)$ and using the second equation (1.8), differentiated with respect to z, and also the relation $\rho W'(\rho) = c_T^2$, where $c_T = \sqrt{\gamma p/\rho}$ is the velocity of sound, we get

$$r^2 = R^2\,(1 - V^2 / c_T^2) \qquad (1.10)$$

This equation defines the curve OO', shown in Fig.1. On this curve, the component of the current density $j_\perp$, perpendicular to the streamlines vanishes. From (1.11), it is clear that this curve connects the point ($r = 0$, $V = c_T$) lying at the narrowest cross-section of the nozzle and the point ($r = R$, $V = 0$). Singular points of the family of lines I = const may lie only on the curve OO', and consequently, are located below the straight line $r = R$ on streamlines with $dr/dz > 0$.

To identify the type of a singular point, we calculate the second derivatives of $I(r,z)$. Considering that

$$W'' = \frac{\gamma - 2}{\rho} W' = \frac{\gamma - 2}{\rho^2} c_T^2 \qquad \gamma \text{ is adiabatic exponent} \qquad (1.11)$$

and using Equations (1.8) to (1.10), we get

$$\frac{\partial^2 I}{\partial r^2} = -8\rho B + \rho^3 r^4 V^2 B'' \qquad (1.12)$$

$$\frac{\partial^2 I}{\partial r \partial z} = -\frac{\rho r V' B}{V}\left(1 - \frac{2V^2}{c_T^2}\right) - \frac{\rho^3}{2} r^5 \frac{V^3}{c_T^2} V' B'' \qquad (1.13)$$

$$\frac{\partial^2 I}{\partial z^2} = -\frac{\rho R^2 V'^2 B}{c_T^2}\left[1 + (\gamma + 2)\frac{V^2}{c_T^2} - \frac{2V^4}{c_T^4}\right] + \frac{\rho^3}{4} r^6 \frac{V^4}{c_T^4} V'^2 B'' \qquad (1.14)$$

The sign of the invariant $\Sigma = I_{rr} I_{zz} - I_{rz}^2$ determines the type of the singular point. Further, restricting ourselves to the case of linear dependence $B(\xi)$, we find that the sign of Σ agrees with the sign of the Expression

$$-1 + 7\frac{V^2}{c_T^2} + 2(\gamma - 2)\frac{V^4}{c_T^4} \qquad (1.15)$$

From this, it follows that if the singular point lies on the part of the curve OO' located below the point

$$r_c = 0.92\,R, \qquad V_c = 0.22\,u \qquad (1.16)$$

then it is elliptic in type; if it lies above r_c , then it is hyperbolic in type (Fig.1).

We note that in the case considered

$$B = b(\xi + c), \quad I = \rho r^2 b \; (\xi + c) \tag{1.17}$$

one of the streamlines $\xi = -c$ will at the same time be a current line. On this line, the magnetic field H_φ vanishes. If we substitute in (1.17) the expressions for ρ and ξ , then we get the equation for the electric current lines

$$V(u^2 - V^2)^{\frac{2}{\gamma-1}} r^2(r^2 - R^2) + C(u^2 - V^2)^{\frac{1}{\gamma-1}} r^2 = \text{const}$$

which may be solved with respect to r^2 .

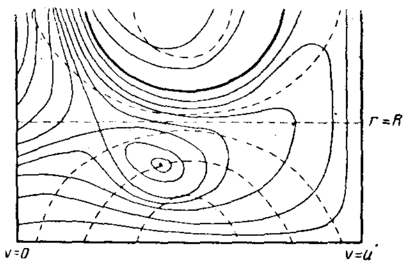

Fig. 2

For negative values of c (Fig. 2) the family of current lines may have two singular points – elliptic and hyperbolic. When $c = 0$, the hyperbolic point moves to the channel inlet (Fig. 3) where $V = 0$, while the elliptic point lies on $r = R/\sqrt{2}$. When $c > 0$, only an elliptic singular point occurs.

In Fig. 2 to 4 electric current lines for a linear dependence $B = b(\xi + c)$ are shown. The dotted lines represent the fluid streamlines, each of which may represent an electrode. The heavy line represents the streamline $\xi = -c$, on which the magnetic field $H\varphi$ change sign. Fig.2 corresponds to $c < 0$; here occur both elliptic and hyperbolic singular points. Fig.3 and 4 correspond to $c = 0$ and $c > 0$, when only elliptic singular points occur.

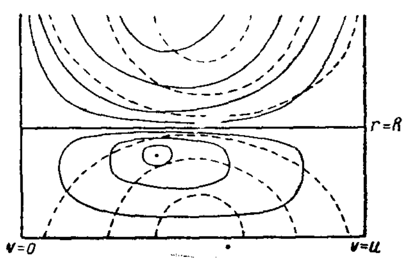

Fig. 3

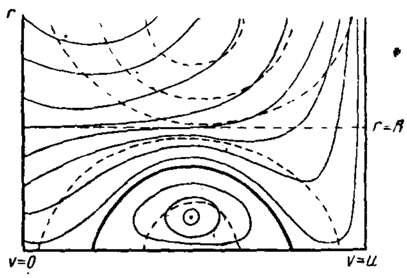

Fig. 4

2. Critical surfaces. The flows, slowly varying along the coordinate z, may be completely calculated in the limiting cases of weak and strong magnetic fields $\beta \gg 1$ and $\beta \ll 1$.

Such calculations, considering only the first derivatives with respect to z , have been carried out in [2 and 3]; there it was shown that in the channels of the corresponding profiles the plasma is accelerated, and the velocity equal to the signal velocity is attained at the narrowest part of the channel. The transition surface (on which the fluid velocity equals the signal velocity) turned out to be inclined toward the side of the fluid motion (curve OA in Fig. 5).

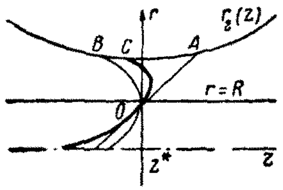

Fig. 5

Nevertheless, taking into accout the terms in the basic equations (1.1) and (1.2) that contain $d^2\xi/dz^2$ and $(d\xi/dz)^2$ results in a bending of the "sonic surface" toward the opposite direction.

We carry out here the corresponding calculations for weak and strong magnetic fields. limiting ourselves to the case U = const and considering in the first approximation the variability of $B(\xi)$.

When $U(\xi)$ = const , the flow is described by Equations

$$\frac{\partial}{\partial r}\frac{1}{\rho r}\frac{\partial\xi}{\partial r}+\frac{\partial}{\partial z}\frac{1}{\rho r}\frac{\partial\xi}{\partial z}+\frac{\rho^2r^3}{2}\frac{dB^2}{d\xi}=0, \qquad W(\rho)+\frac{v^2}{2}+\rho r^2B^2=U \qquad (2.1)$$

Here the velocity components and the magnetic field are

$$v_z=\frac{1}{\rho r}\frac{\partial\xi}{\partial r}, \qquad v_r=-\frac{1}{\rho r}\frac{\partial\xi}{\partial z}, \qquad H_\varphi=\sqrt{4\pi}\,\rho rB \qquad (2.2)$$

As is obvious from (2.1), when B = const , the flow will be irrotational (rot $\mathbf{v}$ = 0). Restricting ourselves to the flows in question, namely those which vary slowly along the z-axis, we seek solutions to the system (2.1) in the form

$$\xi=\xi_0(\varepsilon z, r)+\xi_1(\varepsilon z, r)+\ldots \qquad (\xi_1\sim\varepsilon^2\ll 1) \qquad (2.3)$$

Correspondingly, $v_r\sim\varepsilon$, while $v_z = V + v_{1z}$, where $v_{1z}\sim\varepsilon^2$. Moreover, we shall consider that $B=B_0+B'\xi$, where $B'\xi\ll B_0$, we shall consider everywhere only quantities of first order in B', and reject those terms containing the product of B' with all other small parameters.

In the first approximation, neglecting terms $\sim\varepsilon^2$ in Equation (2.1), we obtain $V = V(z)$.

We shall consider further $V(z)$ as the given velocity on the electrode $r = R$ = const . Solving Equation (2.1), we may find with given accuracy the function $\xi(r,z)$ and from condition $\xi(r,z)$ = const , determine the system of electrodes, corresponding to a given velocity $V(z)$. We confine ourselves here only to the determination of the sonic surfaces. To this end, it suffices to know only the velocity v , i.e. to find the first integral of Equation (2.1).

In the first approximation

$$\xi_0=\int_R^r\rho V(z)\,rdr \qquad (2.4)$$

Substituting this expression into (2.1). we find the correction to the longitudinal velocity, proportional to ε^2.

$$v_{1z}=-\int_R^r dr'\,\frac{\partial}{\partial z}\left[\frac{1}{\rho r'}\int_R^{r'}\frac{\partial}{\partial z}(\rho V)\,rdr\right] \qquad (2.5)$$

The square of the radial velocity v_r is determined to the required accuracy by differentiating (2.4). In order to progress further, it is necessary to know the function $\rho(r,z)$, which is different for the cases of weak ($\beta \gg 1$) and strong ($\beta \ll 1$) magnetic fields. In the zeroth approximation, we have

$$W(\rho)+\frac{V^2}{2}=U, \quad \rho=\rho(z) \quad \text{for } B^2\to 0 \tag{2.6}$$

$$\frac{V^2}{2}+\rho r^2B^2=U, \quad \rho r^2=\frac{U-{}^1/_2V^2}{B^2}\equiv f(z) \quad \text{for } W\to 0 \tag{2.7}$$

Substituting these expressions into (2.4) and (2.5), we find for these cases

$$v_r=-\frac{(\rho V)'}{\rho}\frac{r^2-R^2}{2r}, \qquad v_{1z}=-\frac{1}{4}\left(\frac{(\rho V)'}{\rho}\right)'\left(r^2-R^2-R^2\ln\frac{r^2}{R^2}\right) \tag{2.8}$$

$$v_r=-\frac{(fV)'}{f}r\ln\frac{r}{R}, \qquad v_{1z}=\frac{1}{4}\left(\frac{(fV)'}{f}\right)'\left(r^2-R^2-r^2\ln\frac{r^2}{R^2}\right) \tag{2.9}$$

The transition surface on which the velocity crosses the local signal velocity [4]

$$c_s=\sqrt{c_T{}^2+c_A{}^2} \qquad (c_A^2=H_\varphi{}^2/4\pi\rho) \tag{2.10}$$

is found from Equation $v^2=c_s{}^2$. According to (2.2), we have

$$c_s{}^2=c_T{}^2+\rho r^2B^2\equiv(\gamma-1)\,W+\rho r^2B^2 \tag{2.11}$$

To the required approximation, the velocity is given by Expression

$$v^2=V^2+2Vv_{1z}+v_r{}^2 \tag{2.12}$$

Let u denote the velocity at the exit, where $\rho\to 0(u^2=2U)$, and using (2.1), we represent the equation of the sonic surface for the cases of weak and strong field, respectively, as

$$v^2=\frac{\gamma-1}{\gamma+1}u^2+2\,\frac{2-\gamma}{\gamma+1}\rho r^2B^2, \qquad v^2=\frac{u^2}{3}-2\,\frac{2-\gamma}{3}W(\rho) \tag{2.13}$$

Here, in both expressions, the second term is small, and in it we may substitute ρ from (2.6) or (2.8), respectively.

In the first approximation in ϵ, in the limiting cases of $B^2\to 0$ or $W\to 0$, the sonic surface will be plane and given by Equations

$$V^2(z)=\frac{\gamma-1}{\gamma+1}u^2, \qquad V^2(z)=\frac{1}{3}u^2 \tag{2.14}$$

Terms of order B^2 in the first equation of (2.13) and order W in the second stipulate that this surface (for $r>R$) is inclined toward the side of increasing $V(z)$.

To determine the effect connected with the inclusion of the quantities $\sim\epsilon^2$, it is necessary, on the left-hand side of Equation (2.13), to substitute v^2 from the second equation of (2.1).

We note that according to relations (2.6) and (2.7), for the cases of $\beta\gg 1$ and $\beta\ll 1$ we may write

$$\frac{(\rho V)'}{\rho}=\left(1-\frac{V^2}{c^2{}_T}\right)V', \qquad \frac{(fV)'}{f}=\frac{u^2-3V^2}{u^2-V^2}V' \tag{2.15}$$

These quantities, and consequently, also the radial components of the velocity, vanish on the sonic surface in the zero-th approximation. The latter is obvious from the fact that the sonic surface in the zero-th approximation coincides with the plane of the minimum cross-section of the nozzle.

To obtain the complete expression for the velocity we must also take into account a correction to V, due to variability of $B(\xi)$. According to (2.1), this correction is

$$\delta V = -B_0 B' \int_R^r \rho^2 r^3 dr \tag{2.16}$$

where instead of ρ in the cases of $\beta \gg 1$ and $\beta \ll 1$ we may correspondingly substitute the expressions from (2.6) and (2.7). Thus, the equations of the sonic surfaces assume the form

$$V^2 + \frac{\gamma+1}{2} V'^2\left(r^2 - R^2 - R^2 \ln\frac{r^2}{R^2}\right) = c_T{}^{*2} + 2\frac{2-\gamma}{\gamma+1}\rho r^2 B^2 - \frac{BB'\rho^2 V}{2}(r^4 - R^4) \tag{2.17}$$

$$V^2 - \frac{3}{2} V'^2\left(r^2 - R^2 - r^2 \ln\frac{r^2}{R^2}\right) = c_A{}^{*2} - 2\frac{2-\gamma}{3} W - \frac{B' c_A{}^3}{B^3}\ln\frac{r}{R} \tag{2.18}$$

If the expansion of $V(z)$ is limited to the linear terms, then taking the zero-th approximation sonic surface to be the plane $z = 0$, we find

$$V'z = \frac{2-\gamma}{\gamma+1}\frac{\rho B^2}{c_T{}^2}(r^2 - R^2) + \frac{\gamma+1}{4c_T} V'^2\left(r^2 - R^2 - R^2\ln\frac{r^2}{R^2}\right) + BB'\rho^2\frac{r^4 - R^4}{4} \tag{2.19}$$

$$V'z = -\frac{2-\gamma}{3}\frac{W_R}{c_A}\left[\left(\frac{R^2}{r^2}\right)^{\gamma-1} - 1\right] + \frac{3}{4c_A} V'^2\left(r^2 - R^2 - r^2\ln\frac{r^2}{R^2}\right) + \frac{B' c_A{}^2}{2B^3}\ln\frac{r}{R} \tag{2.20}$$

Here terms $\sim V'^2$ in (2.19) as well as in (2.20) give the curvature of the sonic surface toward the side opposite to the mass flow $(z < 0)$. In fact, expanding functions of r in powers of $(r - R)$, we get

$$V'z = \frac{2-\gamma}{\gamma+1}\frac{2\rho B^2 R}{c_T}(r - R) - \frac{\gamma+1}{2c_T} V'^2 (r - R)^2 + BB'\rho^2 R^3 (r - R) \tag{2.21}$$

$$V'z = \frac{2-\gamma}{3}\frac{2(\gamma-1)W_R}{c_A R}(r - R) - \frac{3}{2c_A} V'^2 (r - R)^2 + \frac{B' c_A^2}{2B^3 R}(r - R) \tag{2.22}$$

As shown by Equations (2.21) and (2.22), the influence of the terms $\sim V'^2$ increases away from the streamline $r = R$. Terms $\sim B'$ partially compensate the main terms linear in $(r - R)$ when $BB' < 0$, and reinforce them when $BB' > 0$.

In the cases of both weak and strong magnetic fields the sonic surfaces have the form schematically shown in Fig.5, where the line OA corresponds to the neglecting of terms $\sim V'^2$, while the line OB shows the shape of the sonic su face for limiting weak or limiting strong fields, when the terms $\sim B^2$ or $\sim W$ (and also $\sim B'$) may be neglected in (2.21) or (2.22). When $B = 0$, (2.21) becomes the well-known expression for the sonic surface in ordinary gas dynamics [5].

3. On the stability of the flow. Restricting ourselves to the zero-th approximation in the small parameter ϵ, we shall consider the stability of a cylindrical plasma jet in a magnetic field having only an azimuthal component $H = H_\varphi$. In a system of coordinates, moving with fluid velocity V, the fluid is at rest, and we may apply the criterion on the absence of convective instability [6]

$$-\frac{d\ln p}{d\ln r} < \frac{4\gamma}{2+\gamma\beta} \qquad \left(\beta \equiv \frac{8\pi p}{H^2}\right) \tag{3.1}$$

Considering the condition of equilibrium along the radius

$$p' = -\frac{H}{4\pi r}(rH)' \tag{3.2}$$

(satisfied in the first approximation in ϵ), the criterion (3.1) may be represented in the form

$$\left(\frac{H^2}{r^2}\right)' + \frac{H^2}{\pi\rho r^3 c_s^2} < 0 \qquad \left(c_s^2 \equiv c_T^2 + \frac{H^2}{4\pi\rho}\right) \tag{3.3}$$

This stability condition (3.3) was obtained in [7] (*) on the basis of an analysis of balance of forces, acting on an annular plasma tube. It is the local stability condition for thin plasma annuli. Violation of this condition results in the displacement of the annuli along the radius. In the present case, they are simultaneously convected together with the flow along the z-axis.

If condition (3.1) is satisfied, then in the absence of helical instability we must require yet another criterion [6]

$$\frac{d\ln p}{d\ln r} < \frac{1}{\beta} \quad \text{or} \quad (rH_\varphi^2)' < 0 \tag{3.4}$$

When (3.4) is violated, helical instability arises with mode $m = 1$ and low pitch. From Expressions (3.3) and (3.4), it follows that for stability with respect to axisymmetric perturbations, it is at least necessary that the derivative $(H^2/r^2)'$ be negative; while for stability with respect to helical perturbations, we must require the derivative $(rH^2)'$ to be negative.

It turns out that in the isentropic case [$S(\xi)$ = const], the criterion (3.3) is expressed only in terms of the "frozen" integral $B(\xi) = H/4\pi\rho r$. In fact, expressing H in terms of B, we may express (3.2) and (3.3) in the form

$$c_s^2\rho' = -\rho^2 B\,(r^2 B)', \qquad c_s^2\,(\rho B^2)' + 4\rho^3 r B^4 < 0 \tag{3.5}$$

where the prime represents differentiation with respect to r. Eliminating ρ, from these, we get

$$dB^2 / dr < 0 \tag{3.6}$$

Since $dB^2/dr = \rho r V\, dB^2/d\xi$ and $V > 0$, then for stability with respect to axisymmetric disturbances it is necessary that $dB^2/d\xi < 0$. In Section 1 flows with a linear dependence $B = b\,(\xi + c)$ were considered. In this case

$$H = \sqrt{4\pi}\,\rho r b\,[{}^1\!/_2\,\rho V\,(r^2 - R^2) + c] \tag{3.7}$$

in which, for the purpose of investigating stability, the slowly varying functions $\rho(z)$ and $V(z)$ occurring, may be taken to be constants. From

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*) See also [1] page 286.

condition (3.6), it follows that stability occurs in the region $\xi < -c$, located in Figs. 2 to 4 below the curve (represented by heavy line) on which $\xi = -c$.

For the case $c = 0$ (cf. Fig.2) condition (3.6) leads to the requirement $r < R$. Criterion (3.4) narrows down the stability region to the interval $^3/_7\, R^2 < r^2 < R^2$, so that the singular point on the current line, lying at $r^2 = {}^1/_2 R^2$, falls into the stable region.

Similarly, in the cases $c \gtrless 0$ condition (3.6) shows that stability occurs in the region $\xi < -c$, or for $b > 0$, in the region negative H_φ.

The coordinates (r, V) of the singular point in the family of current lines I = const are related by

$$r_c{}^2 = \frac{R^2}{2} - \frac{c}{\rho V} \tag{3.8}$$

This relation was obtained from the requirement $\partial I/\partial r = 0$ under the condition that $B(\xi) = b(\xi + c)$. Criterion (3.4) in this case gives

$$r_c^2 > \frac{6}{7}\left(\frac{R^2}{2} - \frac{c}{\rho V}\right) \tag{3.9}$$

i.e. the singular point also falls inside the stable region. However, it lies in the immediate neighborhood of the inner electrode.

We note that in the case of weak magnetic field $H \sim rB(\xi)$, and for B = const, the two stability criteria (3.6) and (3.4) are not satisfied. For strong magnetic field, the case B = const corresponds to stability with respect to helical perturbations, since $H \sim 1/r$. However, the condition (3.6), which is general for weak and for strong fields, is not satisfied.

In the case of limiting strong magnetic field $\beta^2 \ll 1$ (with $B \neq$ const), for a flow that is slowly varying along the x-axis, there exists the integral [2]

$$\rho r^2 B(\xi) = F(z) \tag{3.10}$$

while the streamlines are determined by Expression

$$\int_0^{\xi} \frac{B(\xi)\, d\xi}{\sqrt{U - F(z) B}} = \sqrt{2}\, F(z) \ln \frac{r}{R} \tag{3.11}$$

From (3.10), it follows that $H = \rho r B = r^{-1} F(z)$, and criterion (3.4) holds everywhere. Investigation of (3.11) for $B = b(\xi + c)$ shows that the streamline pattern obtained is similar to that shown in Figs. 2 to 4. Criterion (3.6) also leads to the requirements $\xi < -c$, so that the stability regions lie below the corresponding curves where H changes sign.

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